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# An Operational Method for Abstract Degenerate Evolution Equations of Hyperbolic Type*

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## INTRODUCTION

The chief aim in this work is to study evolution equations of the form

$$\frac{d}{dt}(Mu) + Lu = h, \tag{1}$$

$$M\frac{du}{dt} + Lu = h, \tag{2}$$

where $L$ and $M$ are suitable closed linear operators between complex Banach spaces and $M$ may be degenerate, that is, $M$ has not a bounded inverse. We have already considered problems of type (1) or (2) in some previons papers [see [8, 9], e.g.], but here, unlike those papers, where the parabolic case is mainly investigated, we are developing an abstract theory for hyperbolic problem.

Hence, we shall be able to handle, as an application of (1) or (2), systems of differential equations with $L = L(x, \partial/\partial x)$, a first-order differential operator, and $M = m(x)$, where $m$ can have zeros. Our results on (1) and (2) follow from some quite general statements relative to the operational equations

$$BMu + Lu = h \tag{3}$$

and

$$MBu + Lu = h, \tag{4}$$

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where $B$ is another closed linear operator in the Banach space $E$ (the derivate with respect to $t$ in (1)) and $L, M$ satisfy hypotheses of spectral type entering in various, abstract and/or concrete, differential equations.

In handling (3) and (4), we shall adapt the Da Prato–Grisvard device [6] (see also [7]) of seaking a solution under integral form. This implies, among other things, that no density assumption on the domains of the operators is necessary.

The paper is arranged as follows. In Section 1 we consider Eq. (3) under the assumptions that $L(zM+L)^{-1}$ and $(B-z)^{-1}$ exist for all $z$ in the logarithmic region $\Lambda=\Lambda(a, b)$, where

$$\Lambda=\Lambda(a, b)=\{z\colon \operatorname{Re} z \geqslant a+b\log(1+|z|)\}, \qquad a, b \geqslant 0,$$

and in a neighbourhood of the complementary set of $\Lambda$, respectively, that the commute and their norms in $L(E)$ have a polynomial growth ($L(E)$ denotes the space of all bounded linear operators from $E$ into itself).

We obtain existence (and uniqueness) results under the hypothesis that $h \in D(B^k)$, the domain of $B^k$, $k$ a sufficiently large integer number (time regularity), or $h \in R(T^k)$, the range of $T^k$ (space regularity), where $T=ML^{-1}$ (in fact, we always suppose that $L$ is invertible and $D(L) \subseteq D(M)$).

Section 2 is devoted to Eq. (3), as in Section 1, except the $a$-region

$$U(a)=\{z\colon \operatorname{Re} z \geqslant |\operatorname{Im} z|^a\},\ 0<a<1,$$

replaces $\Lambda(a, b)$.

Now existence of the solution for (3) is proved only for all $h$ in a dense subspace of $R(T^n)$, for a certain integer number (space regularity). In this section we obtain an abstract extension of R. Beals' results in [1], concerning the Cauchy problem connected with (1), $M$ the identity operator. We also refer to [2–4], where there are studied regular (abstract Gevrey spaces-valued) solutions for this problem, with applications to hyperbolic equations with multiple characteristics).

In Section 3, Eq. (4) is considered under both of the preceding spectral hypotheses. Substantially, (4) is reduced to an equation of type (3) by noting that if $L^{-1}h \in D(B)$, then (4) is equivalent to a type-(3) equation $B(L^{-1}M)\,v+v=BL^{-1}h,\ Bu=v$.

In Section 4 we apply what we have previously obtained to abstract differential equations (1) and (2). The section is divided into two parts. In the first one we treat some initial value problems connected with them, and we show that under certain compatibility conditions on the data, they can be transformed into an operational problem of type (3) and (4) in the space $C[0, \tau; X]$ of all strongly continuous functions from $[0, \tau]$ into the Banach space $X$ (in which $L, M$ assume their values) with the sup-norm. As an example, we consider a degenerate hyperbolic system [for a different treatment of it, based upon the theory in [6] under hypotheses of hyperbolic type, see [10]].

In the second part we deal with an equation as $(\sum_{k=0}^{m} A_k\, d^k/dt^k)\, u = f$. It is reduced to a first order equation and then we give some conditions on the operator-pencil $\sum_{k=0}^{m} z^k A_k$ that shall permit us to use what we obtained previously in the first part. We observe that in [5], J. Chazarain obtained necessary and sufficient conditions for the well posedness of the corresponding Cauchy problem in the sense of distributions and of Gevrey-type distributions; they are expressed just in terms of the regions (logarithmic or $a$-region) which we are considering.

## 1. Logarithmic Regions

In this section $L$ and $M$ will denote two closed linear operators from a complex Banach space $F$ into another Banach space $E$ such that $zM + L = P(z)$ has a bounded inverse $P(z)^{-1}$ in the logarithmic region $\Lambda(a, b) = \Lambda$; without loss of generality we shall always suppose that $L$ is invertible and $D(L) \subseteq D(M)$. Before our theorems concerning Eq. (3), we list some assumptions which will be used henceforth.

Assumption H.1. *There are a positive constant $M$ and a non negative integer $m$ such that*

$$\|LP(z)^{-1}; L(E)\| \leqslant M(1 + |z|)^m, \qquad z \in \Lambda.$$

Concerning $B$, we shall suppose that it is a closed linear operator in $E$ satisfying

Assumption H.2. *The resolvent $(B - z)^{-1}$ exists for all complex numbers $z$ in a neighborhood $\Omega$ of $\mathbb{C} \setminus \Lambda$, and there are $M' > 0$, $p \geqslant 0$ such that*

$$\|(B - z)^{-1}; L(E)\| \leqslant M'(1 + |z|)^p, \qquad z \in \Omega.$$

Last, we shall assume that $B$ commutes with the pair of operators $L$, $M$, according to

Assumption H.3. *For all $z' \in \Omega$ and $z'' \in \Lambda$, the commutator $(B - z')^{-1}\, LP(z'')^{-1} - LP(z'')^{-1}\, (B - z')^{-1}$ vanishes.*

The first result we obtain is

Theorem 1.1. *Under Assumptions* H.1–H.3, *Eq.* (3) *has at least one solution for all $h \in D(B^k)$, where $k$ is the smallest integer greater than $m + p + 1$.*

*Proof.* Let $\gamma$ be the boundary of the logarithmic region $\Lambda(a, b)$, oriented clockwise with respect to $\Lambda$; we may assume that $z=0$ lies on the left-hand side of $\gamma$. For all $f \in E$ the integral

$$(2\pi i)^{-1} \int_\gamma z^{-k} P(z)^{-1} (B-z)^{-1} f \, dz = J_k f$$

defines a bounded linear operator $J_k$, in view of H.1, H.2 and the hypothesis on $k$. Here the integral may be viewed either as a generalized Riemann integral or as a Bochner one. Let us prove that $u = J_k B^k h$ satisfies (3). To this end, define

$$I_j f = (2\pi i)^{-1} \int_\gamma z^{-j} (zT+I)^{-1} (B-z)^{-1} f \, dz, \qquad j \geqslant k.$$

Then, it is an easy matter (by a standard technique in vector-valued analytic functions) to see that $u \in D(M)$ and (recall that $T = ML^{-1}$ is a bounded operator in $E$),

$$Mu = TI_k B^k h = B^{-1} h - I_{k+1} B^k h.$$

It follows that $Mu \in D(B)$ and, since

$$\int_\gamma z^{-(k+1)} (zT+I)^{-1} \, dz = 0,$$

we have $BMu = h - Lu$. Q.E.D.

COROLLARY 1.2. *If* H.1–H.3 *hold and* $h \in D(B^\infty) = \bigcap_{j=1}^\infty D(B^j)$, *then the solution* $u = J_k B^k h$ *of* (3) *satisfies* $Mu \in D(B^\infty)$.

It follows, in particular, that if $E = F$ and $(B - z')^{-1}$ commutes with $P(z'')^{-1}$ for all $z'$ and $z''$ in a neighborhood of $\gamma$, then $u \in D(B^\infty)$.

*Remark* 1.3. Theorem 1.1 allows, of course, to handle (3) without any degeneration in $M$, that is, when $M$ coincides with the identity operator in $E = F$, $b = 0$ in the definition of $\Lambda$ and $p = 0$ in Assumption H.2; one then obtains a typically hyperbolic problem. For example, let $1 \leqslant q < \infty$, $0 < \tau < \infty$, $X$ be a Banach space over the complex field. Define the operator $B$ by means of $D(B) = W_0^{1,q}[0, \tau; X]$, the space of all $u \in L^q(0, \tau; X)$ such that the derivative $u' \in L^q(0, \tau; X)$ and $u(0) = 0$, $Bu = u'$.

Then $B$ satisfies H.2, with $p = 0$, in any halfplane $\operatorname{Re} z \leqslant a$.

*Remark* 1.4. If the operator $B$ in H.2 is also densely defined in $E$, and hence $D(B^j)$ is dense in $E$ for all $j \geqslant 1$, then for all $h \in E$ there is a sequence of $h_n \in D(B^k)$, with $k > m + p + 1$, such that $h_n$ converges to $h$ in $E$ as

$n \to \infty$; therefore there exist the strong limits of $B^{-k}Lu_n$ and of $B^{-(k-1)}Mu_n$ as $n \to \infty$, where $u_n = J_k B^k h_n$.

In particular, if $E = F$ and $L$ itself commutes with $B$, we deduce the existence of the strong limit of $B^{-k}u_n$ as $n \to \infty$. So for each $h \in E$ there is a sequence $u_n \in D(L)$ such that $Mu_n \in D(B)$ for all $n = 1, 2, \ldots$, $BMu_n + Lu_n$ converges in $E$ to $h$ as $n \to \infty$ and $B^{-k}u_n$ has a limit in $E$ (as $n \to \infty$). We could say that the sequence $u_n$ defines a "generalized" solution for (3).

Referring to the operator $B$ we have considered in Remark 1.3, $w$ is the strong limit of $B^{-k}u_n$ as $n \to \infty$ if the sequence of the primitives of order $k$ of $u_n = u_n(t)$, vanishing in $t = 0$ together all their derivatives up to order $k-1$, has $w = w(t)$ as a limit in $L^q(0, \tau; X)$.

Next we will examine what happens when one changes to role of the operators $B$ and $T$ in the operational technique we used. If $Lu = v$, then (3) reads $BTv + v = h$; assume for a moment that $T$ has a bounded inverse so we are allowed to apply to the equation $Bw + T^{-1}w = h$ the existence results in [6–8]. Accordingly, we have

$$\begin{aligned} w &= -(2\pi i)^{-1} \int_{\gamma_1} z^{-k}(z - B)^{-1} (T^{-1} + z)^{-1} (-T^{-1})^k (-h)\, dz \\ &= (-1)^k (2\pi i)^{-1} \int_{\gamma_1} z^{-k} T(B - z)^{-1} (zT + I)^{-1} T^{-k} h\, dz, \end{aligned}$$

where $k$ is a certain positive integer and $\gamma_1$ is a suitable unbounded contour in the complex plane. The problem is to give some sense to $T^{-k}h$ when $T$ is not invertible. Now, if $h \in R(T^k)$, $T^{-k}h$ of course doesn't occur in the expression for $w$ and thus we can hope that $u = S_k f$, where

$$S_k = (-1)^k (2\pi i)^{-1} \int_{\gamma_1} z^{-k} L^{-1}(B - z)^{-1} (zT + I)^{-1}\, dz, \quad T^k f = h,$$

is a solution for (3).

In fact, we will prove the following result, which permits the solving of (3) under the hypothesis that $h$ belongs to the range of a certain power of $T$, without any "regularity," as in Theorem 1.1.

THEOREM 1.5. *If Assumptions* H.1–H.3 *hold and the integer $k$ is the smallest one greater than $m + p + 1$, then* (3) *has a solution for any* $h \in R(T^k)$.

*Proof.* Let us specify the contour $\gamma_1$ which occurs in the definition of $S_k : \gamma_1$ shall indicate the path composed from $\operatorname{Re} z = a + b \log(1 + |z|)$ for $|z| \geqslant a_0 > 0$ and $z = a_0 e^{i\phi}$, where $\phi_0 \leqslant \phi \leqslant 2\pi - \phi_0$, $\phi_0 =$

arc $\cos(a_0^{-1}(a+b\log(1+a_0)))$, $a_0$ a suitably chosen constant; $\gamma_1$ is oriented so that $\operatorname{Im} z$ increases along $\gamma_1$. Such a number as $a_0$ exists since $L$ has a bounded inverse. Now we shall quickly show that $u=S_k f$, with $T^k f=h$, solves (3). First, $u\in D(M)$ and

$$Mu=(-1)^{k+1}(2\pi i)^{-1}\int_{\gamma_1} z^{-(k+1)}(B-z)^{-1}(zT+I)^{-1} f\,dz.$$

This implies that $Mu\in D(B)$ and

$$BMu=(-1)^{k+1}(2\pi i)^{-1}\int_{\gamma_1} z^{-(k+1)}(zT+I)^{-1} f\,dz+Lu$$
$$=T^k f+Lu=h+Lu,$$

because

$$\int_{\gamma_1} z^{-(k+1)}(zT+I)^{-1} f\,dz=-\int_{\Gamma_\varepsilon} z^{-(k+1)}(zT+I)^{-1} f\,dz,$$

where $\Gamma_\varepsilon$ denotes the counterclockwise oriented circumference $|z|=\varepsilon$, with a small radius $\varepsilon$.

On the other hand, it is simple routine to verify that this last expression coincides with $(2\pi i)(-1)^{k+1}T^k f$. Q.E.D.

*Remark* 1.6. If $M=I$, Theorem 1.5 affirms that (3) is solvable for each $h\in D(L^k)$.

*Remark* 1.7. If $h\in\overline{R(T^k)}$, the closure of $R(T^k)$ in $E$, and hence $h$ is the strong limit of $h_n=T^k f_n$ as $n\to\infty$, in view of Theorem 1.5, (3) has one solution $u_n=S_k f_n$ corresponding to the right-hand side $h_n$. This implies that there exists $u_n\in D(L)$ such that $BM_u+Lu_n\to h$ in $E$ and $T^kLu_n$ has a limit in $E$ as $n\to\infty$.

Note that if $k=1$, $M=I$ and $D(L)$ is dense in $E=F$, one should obtain a strong solution for (3) in the sense of the definition in [6].

*Remark* 1.8. If $h\in\bigcap_{j=1}^{\infty} R(T^j)=R(T^\infty)$ in Theorem 1.5, then $Lu\in R(T^\infty)$, too, for $h=T^j f_j$, $j>k$, implies $h=T^k\,(T^{j-k}f_j)$ and hence $Lu=T^{j-k}S_k f_j$. In particular, if $M=I$, from $h\in D(L^\infty)$ it follows that $u\in D(L^\infty)$. Next result provides uniqueness of the solution for problem (3); it reads:

THEOREM 1.9. *Under* H.1–H.3, *Problem* (3) *has at most one solution.*

*Proof.* It is enough to show that if $BTv+v=0$, then $v=0$. In fact, if $\gamma_1$ indicates the path introduced in the proof of Theorem 1.5 and $k$ is chosen in such a way that the integrals we shall consider converge, then

$$0 = (2\pi)^{-1} \int_{\gamma_1} z^{-k}(zT+I)^{-1} (B-z)^{-1} \{BTv + v\} \, dz$$

$$= (2\pi i)^{-1} \int_{\gamma_1} z^{-k} B(B-z)^{-1} (zT+I)^{-1} Tv \, dz$$

$$+ (2\pi i)^{-1} \int_{\gamma_1} z^{-k}(zT+I)^{-1} (B-z)^{-1} v \, dz$$

$$= (\mathrm{i}) + (\mathrm{ii}).$$

By writing $Tv$ as $z^{-1}(zT+I-I)\, v$, $z \neq 0$, we see that

$$(\mathrm{i}) = -(2\pi i)^{-1} \int_{\gamma_1} z^{-(k+1)} B(B-z)^{-1} (zT+I)^{-1} v \, dz$$

$$= -(2\pi i)^{-1} \int_{\gamma_1} z^{-(k+1)}(zT+I)^{-1} v \, dz - (\mathrm{ii}).$$

The first addendum in the last expression is nothing else that $(-1)^k T^k v$. This implies that $T^k v = 0$. Since $BTv + v = 0$, application of $T^{k-1}$ to both members of this equation leads to $T^{k-1} v = 0$ and so on, until $Tv = v = 0$ is obtained. Q.E.D.

## 2. Regions of Type $a$

Definition 2.1. Let $0 < a < 1$; we shall say that the domain $U_a = \{z: z \in \mathbb{C},\ \mathrm{Re}\, z \geqslant |\mathrm{Im}\, z|^a\}$ is a region of type $a$ or an $a$-region.

If $b > 0$, let us define $U(a, b) = U_a \cup \{z: |z| \leqslant b\}$ and denote by $\Gamma$ the contour

$$\Gamma = \{z;\ |z| = b/2,\ \mathrm{Re}\, z < x_0\} \cup \{z: |z| \geqslant b/2,\ \mathrm{Re}\, z = |\mathrm{Im}\, z|^a\}$$

oriented from the lower to the upper halfplane. Here $x_0$ is the unique positive solution of $x^2 + x^{2/a} = b^2/4$. We further need the following assumptions:

Assumption H.4. *The closed linear operators L, M from F into E satisfy the conditions in* H.1, *with* $U(a, b)$ *replacing* $\Lambda(a, b)$.

Assumption H.5. *B is a closed linear operator from E into itself such* *that* $(B-z)^{-1}$ *exists for all* $z$ *in an open set strictly containing* $\mathbb{C}\setminus U(a,b)$ *where holds the estimate*

$$\|(B-z)^{-1}; L(E)\| \leqslant C\exp(|z|^a), \qquad z\in\Gamma,\ |z| \text{ large},$$

($C$ *is a positive constant*).

As we said in the Introduction, we will develop an idea of R. Beals [1] in order to solve (3) in this case. Given $\varepsilon>0$, let $h_\varepsilon(z)$ the branch of $\exp(-\varepsilon(-z)^g)$, $z\notin U_a$, where $a<g<1$, which is positive for $z<0$. One easily recognizes that $|h_\varepsilon(z)|\leqslant\exp(-d|z|^g)$ for all $z\in\Gamma$, $|z|$ large, and this implies that $u_\varepsilon=\mathcal{T}_\varepsilon f$ is well defined for each $f\in E$, where

$$\mathcal{T}_\varepsilon=(2\pi i)^{-1}\int_\Gamma h_\varepsilon(z)\,P(z)^{-1}\,(B-z)^{-1}\,dz.$$

Of course we are going to modify suitably the technique which let us to obtain Theorem 1.5. We shall prove

THEOREM 2.1. *Under Assumptions* H.3–H.5, *problem* (3) *has at least one solution for all* $h$ *in a dense subspace of* $R(T^{m+1})$.

*Proof.* We shall show that one solution for (3) exists if $h$ has a particular form; then, we shall see that such $h$'s form a dense subspace in $R(T^{m+1})$. It is a simple matter to verify that if $f\in E$, $u_\varepsilon=\mathcal{T}_\varepsilon f\in D(M)$, and

$$Mu_\varepsilon=-(2\pi i)^{-1}\int_\Gamma z^{-1}h_\varepsilon(z)(zT+I)^{-1}\,(B-z)^{-1}f\,dz.$$

Furthermore, $u_\varepsilon\in D(L)$, $Mu_\varepsilon\in D(B)$, with

$$BMu_\varepsilon+Lu_\varepsilon=-(2\pi i)^{-1}\int_\Gamma z^{-1}h_\varepsilon(z)(zT+I)^{-1}f\,dz=J_\varepsilon f.$$

And thus, (3) has one solution for any $h$ of the type $h=J_\varepsilon f$ for a certain $f\in E$ and $\varepsilon>0$. We now are going to prove that if $f\in R(T^{m+1})$, then there exists the strong limit of $J_\varepsilon f$, as $\varepsilon\downarrow 0$, and it coincides with $f$ itself. In order to see this, let $f=T^{m+1}f_1$, $f_1\in E$. Then

$$\begin{aligned}
J_\varepsilon f&=-(2\pi i)^{-1}\int_\Gamma z^{-1}h_\varepsilon(z)(zT+I)^{-1}\,T^{m+1}f_1\,dz\\
&=\cdots=(2\pi i)^{-1}\int_\Gamma z^{-2}h_\varepsilon(z)(zT+I)^{-1}\,T^m f_1\,dz\\
&=(-1)^m\,(2\pi i)^{-1}\int_\Gamma z^{-(m+2)}h_\varepsilon(z)(zT+I)^{-1}f_1\,dz.
\end{aligned}$$

Since $|h_\varepsilon(z)| \leqslant C'$ for all $\varepsilon \in (0, 1]$ and $z \in \Gamma$, Lebesgue theorem implies that $J_\varepsilon f$ converges in $E$, as $\varepsilon \downarrow 0$, to $(-1)^m (2\pi i)^{-1} \int_\Gamma z^{-(m+2)}(zT+I)^{-1} f_1\, dz$. On the other hand we easily recognize that the last integral coincides with $-\int_{\Gamma_\delta} z^{-(m+2)}(zT+I)^{-1} f_1\, dz$, where $\Gamma_\delta$ is the counterclockwise oriented circumference $|z| = \delta$. A trivial calculation gives the desired result. Q.E.D.

Uniqueness for the solution of (3) is ensured by

THEOREM 2.2. *If* H.3–H.5 *hold, then* (3) *has almost one solution.*

*Proof.* We need modify slightly the proof of Theorem 1.9. In fact, if $v$ satisfies $BTv + v = 0$, then

$$\begin{aligned} 0 &= (2\pi i)^{-1} \left\{ \int_\Gamma h_\varepsilon(z)(zT+I)^{-1} (B-z)^{-1} BTv\, dz \right. \\ &\quad \left. + \int_\Gamma h_\varepsilon(z)(zT+1)^{-1} (B-z)^{-1} v\, dz \right. \\ &= (1) + (2). \end{aligned}$$

We immediately see [note that we make use of H.3] that

$$(1) = -(2\pi i)^{-1} \int_\Gamma z^{-1} h_\varepsilon(z)\, B(B-z)^{-1} (zT+I)^{-1} v\, dz = J_\varepsilon v - (2).$$

Hence $J_\varepsilon v = 0$ and also $TJ_\varepsilon v = 0$. But this implies

$$\begin{aligned} 0 &= (2\pi i)^{-1} \int_\Gamma z^{-2} h_\varepsilon(zT+I-I)(zT+I)^{-1} v\, dz \\ &= -(2\pi i)^{-1} \int_\Gamma z^{-2} h_\varepsilon(z)(zT+I)^{-1} v\, dz. \end{aligned}$$

Repeating the same argument, we obtain

$$0 = (2\pi i)^{-1} \int_\Gamma z^{-(m+2)} h_\varepsilon(z)(zT+I)^{-1} v\, dz,$$

for all $\varepsilon > 0$. Letting $\varepsilon \downarrow 0$, we have $T^{m+1} v = 0$. We can then conclude, as in the proof of Theorem 1.9, that $v = 0$. Q.E.D.

## 3. THE EQUATION $MBu + Lu = h$

In this section we shall be concerned with Eq. (4), where $B$ is a closed linear operator from $E$ into itself, $L$ and $M$ are closed linear operators from

$E$ into $F$, $h$ is a given element of $F$ and $u$ is the unknown. Of course, $E$ and $F$ are complex Banach spaces.

We have already pointed out in the Introduction in what manner (4) can be reduced to a problem of type (3). Since there are troubles with the closure of the operator $L^{-1}M$ in the space $E$, we are compelled to find a new space in which to consider the reduced equation. It turns out that $D(L)$ is the right space if we slightly modify the preceding assumptions. We shall see that they often occur in the applications.

ASSUMPTION H.6. *The operator $zM + L = P(z)$ has a bounded inverse for all $z \in \Lambda(a, b) = \Lambda$ and*

$$\|LP(z)^{-1}; L(F)\| \leqslant C(1 + |z|)^m, \qquad z \in \Lambda,$$

*where $C > 0$ and $m \geqslant 0$.*

ASSUMPTION H.7. *$D(L)$ is a subspace invariant under $(B - z)^{-1}$ for all $z$ in a neighbourhood $\Omega$ of $\mathbb{C} \setminus \Lambda$, with*

$$\|(B - z)^{-1}; L(D(L))\| \leqslant C'(1 + |z|)^p,$$

*with $C' > 0$ and $p \geqslant 0$.*

ASSUMPTION H.8. *For all $z' \in \Lambda$, $z'' \in \Omega$ and $f \in D(L)$, we have*

$$P(z')^{-1} L(B - z'')^{-1} f = (B - z'')^{-1} P(z')^{-1} Lf.$$

Now we are ready to establish existence and uniqueness results for (4) in the case of a logarithmic region.

THEOREM 3.1. *Assume* H.6–H.8. *If $L^{-1}h \in D(B^k)$, where $k$ is the smallest integer greater than $m + p + 2$, then* (4) *has at least one solution.*

*Proof.* According to the previous reduction, we seek a solution for (4) under the form

$$u = (2\pi i)^{-1} \int_\gamma z^{-(k-1)} B^{-1} P(z)^{-1} (B - z)^{-1} B^k L^{-1} h \, dz,$$

where $\gamma$ is the contour that we have used in the proof of Theorem 1.1. In fact, we have

$$MBu = h - (2\pi i)^{-1} \int_\gamma z^{-k} LP(z)^{-1} L(B - z)^{-1} B^k L^{-1} h \, dz.$$

Now, H.8 implies that we can express $P(z)^{-1} L(B-z)^{-1}$ on $D(L)$ as

$$B^{-1}P(z)^{-1} LB(B-z)^{-1} = B^{-1}P(z)^{-1} L + zB^{-1}P(z)^{-1} L(B-z)^{-1};$$

hence, last addendum in the right hand side of the expression for $MBu$ coincides with $Lu$. Q.E.D.

COROLLARY 3.2. *Suppose $E=F$ and*

$$\|P(z)^{-1}; L(E)\| \leqslant C''(1+|z|)^r, \qquad z \in \Lambda,$$

*with $C''>0$ and $r \geqslant 0$. Assume also* H.6, H.7. *If $B^{-1}$ commutes with $L^{-1}$ and $T(=ML^{-1})$, then* (4) *has one solution for each $h \in D(B^k)$, where $k > \max\{m, r\} + p + 1$.*

*Proof.* The $u$ in the proof of Theorem 3.1 can be put under the form

$$u = (2\pi i)^{-1} \int_\gamma z^{-k} P(z)^{-1}(B-z)^{-1} B^k h \, dz.$$

We have $u \in D(B)$ and, since $P(z)^{-1} = L^{-1}(zT+I)^{-1}$,

$$Bu = (2\pi i)^{-1} \int_\gamma z^{-k+1} P(z)^{-1} (B-z)^{-1} B^k h \, dz.$$

It follows that

$$MBu = (2\pi i)^{-1} \int_\gamma z^{-k}(P(z)-L) P(z)^{-1} (B-z)^{-1} B^k h \, dz = h - Lu.$$

Q.E.D.

The result for (4) corresponding to Theorem 1.5 reads

THEOREM 3.3. *Under Assumptions* H6–H8, *problem* (4) *has one solution for all $h \in R(T^{k+1})$, where $k > m+p+1$.*

*Proof.* We can suppose $h = T^k Mf$, with $f \in D(L)$. We shall prove that

$$u = (-1)^{k+1} (2\pi i)^{-1} \int_{\gamma_1} z^{-(k+1)}(B-z)^{-1} P(z)^{-1} Lf \, dz,$$

where $\gamma_1$ is the path we introduced in the proof of Theorem 1.5, solves (4). First, $u \in D(B)$ with

$$Bu = (-1)^{k+1}(2\pi i)^{-1} \int_{\gamma_1} z^{-(k+1)} P(z)^{-1} Lf\,dz$$
$$+ (-1)^{k+1}(2\pi i)^{-1} \int_{\gamma_1} z^{-k}(B-z)^{-1} P(z)^{-1} Lf\,dz.$$

Furthermore, in view of H.8, $Bu \in D(M)$ and

$$BMu = (-1)^{k+1}(2\pi i)^{-1} \int_{\gamma_1} z^{-(k+1)} T(zT+I)^{-1} Lf\,dz$$
$$+ (-1)^{k+1}(2\pi i)^{-1} \int_{\gamma_1} z^{-k} T(zT+I)^{-1} L(B-z)^{-1} f\,dz$$
$$= (\mathrm{i}) + (\mathrm{ii}).$$

Obviously, by H.8 again,

$$(\mathrm{ii}) = (-1)^{k}(2\pi i)^{-1} \int_{\gamma_1} z^{-(k+1)} LP(z)^{-1} L(B-z)^{-1} f\,dz = -Lu.$$

As for (i), it coincides with $(-1)^k (2\pi i)^{-1} \int_{|z|=\rho} z^{-(k+1)} T(zT+I)^{-1} Lf\,dz$, where $\rho > 0$ is sufficiently small and thus we easily see that $(\mathrm{i}) = T^{k+1} Lf = h$. Q.E.D.

Uniqueness is ensured by

THEOREM 3.4. *If* H.6–H.8 *are satisfied, then Problem* (4) *has at most one solution.*

*Proof.* Suppose $MBu + Lu = 0$, that is, $L^{-1}MBu + u = 0$, and hence $BNBu + Bu = 0$, with $N = L^{-1}M$. Consider $N$ as a bounded operator from $D(L)$ into itself. Our assumption H.6 implies that for all $f \in D(L)$ we have

$$\|(zN+I)^{-1} f; D(L)\| = \|LP(z)^{-1} Lf; F\| \leqslant C(1+|z|)^m \|f; D(L)\|;$$

further,

$$(B-z'')^{-1}(z'N+I)^{-1} f = (B-z'')^{-1} P(z')^{-1} Lf$$
$$= (z'N+I)^{-1}(B-z'')^{-1} f,$$

in virtue of H.8.

Hence, H.7 and Theorem 1.9 applied to $BNv + v = 0$, imply $Bu = u = 0$.
Q.E.D.

To treat (4) under hypotheses analogues to the ones in Section 2, we again need some "ad hoc" definitions.

For brevity, we shall say that Assumption H.6′–H.8′ hold if H.6–H.8, respectively, are satisfied, with $U(a, b)$ [see Definition 2.1] instead of $\Lambda(a, b)$. H.7 is replaced by

ASSUMPTION H.7′. *$D(L)$ is a subspace of $E$ invariant under $(B-z)^{-1}$ for all $z$ in an open neighbourhood of $\mathbb{C} \setminus U(a, b)$ and*

$$\|(B-z)^{-1}; L(D(L))\| \leqslant C \exp(|z|^a),$$

*for each $z \in \Gamma$, with large $|z|$, where $C$ is a positive constant and $\Gamma$ the boundary of $U(a, b)$.*

We will directly show

THEOREM 3.5. *Under Assumption* H.6′–H.8′, (4) *has one solution for all $h$ in a dense subspace of $R(T^{m+1})$.*

*Proof.* Let

$$u_\varepsilon = -(2\pi i)^{-1} \int_\Gamma z^{-1} h_\varepsilon(z)(B-z)^{-1} P(z)^{-1} Lf\, dz,$$

where $\varepsilon > 0$ and $f \in D(L)$. The integral converges, as one easily sees by expressing $(B-z)^{-1} P(z)^{-1}$ as $L^{-1}L(B-z)^{-1} L^{-1}LP(z)^{-1}$ and using H.6′ and H.7′. H.8′ furnishes $u_\varepsilon \in D(B)$ and

$$\begin{aligned} Bu_\varepsilon &= -(2\pi i)^{-1} \left\{ \int_\Gamma z^{-1} h_\varepsilon(z)\, P(z)^{-1}\, Lf\, dz \right. \\ &\qquad \left. + \int_\Gamma h_\varepsilon(z)\, P(z)^{-1}\, L(B-z)^{-1} f\, dz \right\} \\ MBu_\varepsilon &= (2\pi i)^{-1} \left\{ \int_\Gamma z^{-2} h_\varepsilon(z)\, LP(z)^{-1}\, Lf\, dz \right. \\ &\qquad \left. + \int_\Gamma z^{-1} h_\varepsilon(z)\, LP(z)^{-1}\, L(B-z)^{-1} f\, dz \right\} \\ &= -Lu_\varepsilon + J'_\varepsilon f, \end{aligned}$$

where $J'_\varepsilon f = (2\pi i)^{-1} \int_\Gamma z^{-2} h_\varepsilon(z)(zT+I)^{-1}\, Lf\, dz$, $f \in D(L)$. In particular, if $f = (L^{-1}M)^m g$, with $g \in D(L)$, then

$$J'_\varepsilon f = (2\pi i)^{-1} \int_\Gamma z^{-2} h_\varepsilon(z)(zT+I)^{-1} M(L^{-1}M)^{m-1} g\, dz$$

$$= (-1)^j (2\pi i)^{-1} \int_\Gamma z^{-(2+j)} h_\varepsilon(z)(zT+I)^{-1} L(L^{-1}M)^{m-j} g\, dz$$

for $j = 0, 1, ..., m$. Hence, for $j = m$,

$$J'_\varepsilon f = (-1)^m (2\pi i)^{-1} \int_\Gamma z^{-(m+2)} h_\varepsilon(z)(zT+I)^{-1} Lg\, dz,$$

and it is immediate to verify that then $J'_\varepsilon f$ converges to $T^{m+1}Lg$ as $\varepsilon \downarrow 0$. Since $R(L) = F$, this prove our statement. Q.E.D.

By means of an argument similar to the one we have used in proving Theorem 3.4, Theorem 2.2 yields

THEOREM 3.6. *If* H.6′–H.8′ *hold, then* (4) *has at most one solution.*

## 4. APPLICATIONS TO ABSTRACT DIFFERENTIAL EQUATIONS

*Applications* 1

In the study of initial-value problems connected with abstract differential equations, we will confine our investigations in searching for "true" solutions of the problem

$$\begin{gathered} \frac{d}{dt}(A_1 u(t)) + A_0 u(t) = f(t), \quad 0 \leqslant T \leqslant \tau, \\ \lim_{t \downarrow 0} \|A_1 u(t) - w_0; X\| = 0, \end{gathered} \tag{5}$$

where $f$ is a strongly continuous from $[0, \tau]$ into the complex Banach space $X$, $A_i$, $i = 0, 1$, is a closed linear operator from the (complex) Banach space $Y$ into $X$, with $D(A_1) \subseteq D(A_0)$ and $A_0$ invertible.

We at once explain what we intend as one solution for (5).

DEFINITION 4.1. A function $u$ from $[0, \tau]$ into $D(A_0)$ such that $t \to A_0 u(t) \in C[0, \tau; X]$, $t \to A_1 u(t) \in C^{(1)}[0, \tau; X]$ and (5) is satisfied, shall be said to be a Strict solution for (5).

According usual notations, for $m = 1, 2, ..., C^{(m)}[0, \tau; X]$ denotes the space of all $X$-valued $m$-times continuously strongly differentiable functions on $[0, \tau]$.

*Remark* 4.2. If $X=Y$, $A_0=A$ and $A_1$ denotes the embedding operator of $D(A)$ into $X$, then (5) reads

$$\begin{aligned} &u'(t)+Au(t)=f(t), \quad 0\leqslant t\leqslant \tau, \\ &\lim_{t\downarrow 0} \|u(t)-w_0; X\|=0, \end{aligned} \tag{6}$$

the usual Cauchy problem.

In describing our results we shall need the following assumption:

ASSUMPTION H.9. *The operator $zA_1+A_0$ has a bounded inverse for all $z\in\Lambda(a,b)$ and*

$$\|A_0(zA_1+A_0)^{-1}; L(X)\|\leqslant C(1+|z|)^m, \qquad z\in\Lambda(a,b),$$

*with some positive constants C and m.*

Let $A_1A_0^{-1}=S$; if $w_0\in A_1(D(A_0))=\{A_1x: x\in D(A_0)\}$, and hence $w_0=Sv_0$, $v_0\in X$, then $A_0u=v$ transforms (5) into the equivalent problem

$$\begin{aligned} &d(Sv(t))/dt+v(t)=f(t), \quad 0\leqslant t\leqslant \tau, \\ &\lim_{t\downarrow 0} \|S(v(t)-v_0); X\|=0. \end{aligned} \tag{7}$$

Define $z(t)=v(t)-\sum_{j=0}^{r} (t^j/j!)\, v_j$, where $v_1, ..., v_r\in X$ and the positive integer $r$ shall be fixed later on. Then (7) is in turn reduced to

$$\begin{aligned} &d(Sz(t))/dt+z(t)=h(t), \quad 0\leqslant t\leqslant \tau, \\ &\lim_{t\downarrow 0} \|Sz(t); X\|=0, \end{aligned} \tag{8}$$

with $h(t)=f(t)-\sum_{j=0}^{r-1} (t^j/j!)(Sv_{j+1}+v_j)-(t^r/r!)\, v_r$. That being stated, we are now ready to prove

THEOREM 4.3. *Assume* H.9 *and let $p=b\tau$; let $k$ be the smallest positive integer $>m+p+1$. If $f\in C^{(k)}[0,\tau; X]$, then* (5) *has a unique strict solution on $[0,\tau]$ for all $w_0$ of the type*

$$w_0=\sum_{j=0}^{k-1} (-1)^j S^{j+1}(f^{(j)}(0))+w,$$

*where $w\in R(S^{k+1})$.*

*Proof.* We want to apply Theorem 1.1. To this end, we take $C[0,\tau; X]$

as $E$ and $D(B)=\{u\in C^{(1)}[0,\tau;X]\colon u(0)=0\}$, $Bu=u'$. It is immediate to see that

$$\|(B-z)^{-1};L(E)\|\leqslant M(1+|z|)^p,$$

for all $z$ in a neighbourhood of $\mathbb{C}\setminus A(a,b)$. If we choose $r=k$ in the definition of $z(t)$, then Theorems 1.1 and 1.9 state that (8), and hence (5), has a unique strict solution if

$$f^{(j)}(0)-(Sv_{j+1}+v_j)=0,$$

for $j=0,1,\dots,k-1$. Let $v_k=\tilde{v}$. Then $v_{k-1}=f^{(k-1)}(0)-S\tilde{v}$, and successively $v_{k-j}=\sum_{i=0}^{j-1}(-1)^i\,S^if^{(k-j+1)}(0)+(-1)^j\,S^j\tilde{v}$ for $j=2,\dots,k$. In particular, $v_0=\sum_{j=0}^{k-1}(-1)^i\,S^if^{(i)}(0)+(-1)^k\,S^k\tilde{v}$, and also

$$w_0=\sum_{j=0}^{k-1}(-1)^j\,S^{j+1}f^{(j)}(0)+w\qquad\text{with}\quad w\in R(S^{k+1}).\qquad\text{Q.E.D.}$$

*Remark* 4.4. If $A$ is a closed linear operator in the Banach space $X$ such that $z+A$ has a bounded inverse for all $z\in A(a,b)$, and

$$\|(z+A)^{-1};L(X)\|\leqslant C(1+|z|)^{m-1},\qquad z\in A(a,b),\tag{9}$$

for certain constants $C>0$ and $m\geqslant 1$, then the equations

$$f^{(j)}(0)-(A^{-1}v_{j+1}+v_j)=0,\qquad j=0,1,\dots,k-1,$$

could be solved from $j=0$ on, if $f^{(j)}(0)-v_j\in D(A)$ for $j=0,1,\dots,k-1$. Therefore, problem (6) shall have a strict solution if $f^{(j)}(0)-v_j\in D(A)$, $j=0,1,\dots,k-1$, where $v_i=A[f^{(i-1)}(0)-v_{i-1}]$ for $i=1,2,\dots,k$.

*Remark* 4.5. If $f(t)$ vanishes identically on $[0,\tau]$ then the homogeneous problem (5) (resp. (6)), has a unique strict solution for each $w_0\in R(S^{k+1})$ (resp. for all $w_0\in D(A^{k+1})$).

Application of Theorems 1.5 and 1.9 yields the following statements, where no time regularity is assumed for $f(t)$; it is replaced by a range condition: $f=S^kg$. Exactly,

THEOREM 4.6. *Let us assume that* H.9 *holds and let k the smallest integer* $>m+p+1$, $p=b\tau$. *Then problem* (5) *has a unique strict solution on* $[0,\tau]$ *for each* $w_0\in R(S^{k+1})$ *and any* $f$ *of the type* $f(t)=S^kg(t)$, *where* $g\in C[0,\tau;X]$.

*Proof.* It is enough to write $w_0 = Sv_0$, $v(t) = v_0 + w(t)$ and to transform (5) into

$$d(Sw(t))/dt + w(t) = f(t) - v_0, \quad 0 \leqslant t \leqslant \tau,$$
$$\lim_{t \downarrow 0} \|Sw(t); X\| = 0.$$

To such a problem one applies the afore-said theorems with $E = C[0, \tau; X]$. Q.E.D.

COROLLARY 4.7. *If Assumption* (9) *is satisfied and k is chosen as in Theorem* 4.6, *then* (6) *has a unique strict solution for all* $u_0 \in D(A^{k+1})$ *and all f strongly continuous from* $[0, \tau]$ *into* $D(A^k)$.

*Remark* 4.8. It is to be noted that, unlike Theorem 4.3, Theorem 4.6 requires no link between the non homogeneous part $f$ and the initial condition $w_0$ in problem (5).

We now will give some results relative to regions of type $a$; that is, we shall suppose that $A_0$, $A_1$ are closed linear operators from $Y$ into $X$ satisfying H.9 in the region $U_a$ (and hence, in $U(a, b)$).

Let $J_\varepsilon = -(2\pi i)^{-1} \int_\Gamma z^{-1} h_\varepsilon(z)\, A_1(zA_1 + A_0)^{-1}\, dz$, where $\varepsilon > 0$ and $\Gamma$ is the contour we introduced in Section 2. Then the same proof of Theorem 2.1 allows us to deduce that if $f(t) = S^{m+1} g(t)$, $0 \leqslant t \leqslant \tau$, $g \in C[0, \tau; X]$, $S = A_1 A_0^{-1}$, and $w_0 = S^{m+2} u_0$, $u_0 \in X$, then for any $\varepsilon > 0$ there is a unique strict solution $w_\varepsilon = w_\varepsilon(t)$ for the problem

$$d(Sw_\varepsilon(t))/dt + w_\varepsilon(t) = J_\varepsilon[f(t) - S^{m+1} u_0], \quad 0 \leqslant t \leqslant \tau,$$
$$\lim_{t \downarrow 0} Sw_\varepsilon(t) = 0.$$

Therefore, $z_\varepsilon(t) = w_\varepsilon(t) + J_\varepsilon S^{m+1} u_0$ fulfils

$$d(Sz_\varepsilon(t))/dt + z_\varepsilon(t) = J_\varepsilon f(t), \quad 0 \leqslant t \leqslant \tau,$$
$$\lim_{t \downarrow 0} Sz_\varepsilon(t) = J_\varepsilon w_0.$$

It suffices, in fact, to take $E = C[0, \tau; X]$ in that Theorem, with the obvious definitions of $B$, $L$, and $M$.

From Theorem 2.1 we also deduce that

$$\|J_\varepsilon f - f; C[0, \tau; X]\|, \qquad \|J_\varepsilon w_0 - w_0; X\|$$

converge to zero as $\varepsilon \downarrow 0$. And thus we can establish

THEOREM 4.9. *If* H.9 *holds on* $U_a$, *then problem* (5) *has a unique strict solution for each* $f$ *and* $w_0$ *belonging to two subspaces everywhere dense in* $S^{m+1}(C[0, \tau; X])$ *and in* $S^{m+2}(X)$, *respectively.*

COROLLARY 4.10. *Under Assumption* (9) *relative to* $z \in U_a$, *problem* (6) *has a unique strict solution on* $[0, \tau]$ *for each* $f$ *and* $w_0$ *in some dense subspaces of* $C[0, \tau; D(A^{m+1})]$ *and* $D(A^{m+2})$, *respectively.*

In this way we have obtained the result in [1] as said in the Introduction.

*Remark* 4.11. Let $B_i$, $i = 0, 1$, be a closed linear operator from a Banach space $Y$ into another Banach space $X$. Given a strongly continuous function $f$ from $[0, \tau]$ into $X$ and $u_0 \in Y$, we shall say that $u = u(\cdot)$ is a strict solution for the problem

$$B_1 u'(t) + B_0 u(t) = f(t), \quad 0 \leqslant t \leqslant \tau, \qquad u(0) = u_0, \tag{10}$$

if $u \in C^{(1)}[0, \tau; Y]$, $u(t) \in D(B_0)$, and $u'(t) \in D(B_1)$ for all $t \in [0, \tau]$, $B_0 u(\cdot) \in C[0, \tau; X]$ and (10) holds.

Note that necessarily $u_0 \in D(B_0)$. Let $r$ be a positive integer and let $u_1, ..., u_r$ be certain elements in $Y$. If one defines $u(t) - \sum_{i=0}^{r} (t^i/i!)\, u_i = z(t)$, then (10) takes the form

$$B_1 z'(t) + B_0 z(t) = h(t), \quad 0 \leqslant t \leqslant \tau, \qquad z(0) = 0,$$

where $h(t) = f(t) - \sum_{j=0}^{r-1} (t^j/j!)\{B_0 u_j + B_1 u_{j+1}\} - (t^r/r!)\, B_0 u_r$. Hence, application of Theorems 3.1 and 3.4 in a first time and of Theorem 3.3, 3.4 in a second one, leads to

THEOREM 4.12. *Suppose that* $B_0$, $B_1$ *satisfy a condition of type* H.9 (*with* $B_i$ *instead of* $A_i$). *Let* $p = b\tau$. *If* $f \in C^{(k)}[0, \tau; X]$, *where* $k$ *is the smallest integer number* $> m + p + 2$, *then* (10) *has a unique strict solution for all* $u_0 \in D(B_0)$ *such that*

$$B_0 u_0 - \sum_{j=0}^{k-1} (-1)^j (B_1 B_0^{-1})^j f^{(j)}(0) \in R((B_1 B_0^{-1})^k).$$

THEOREM 4.13. *Under the same assumptions as in Theorem* 4.12, (10) *has a unique strict solution for all* $f$ *and* $u_0 \in D(B_0)$ *such that* $f(t) = (B_1 B_0^{-1})^k g(t)$, $B_0 u_0 \in R((B_1 B_0^{-1})^k)$, *where* $k$ *is the smallest integer* $> m + p + 2$.

It suffices, in fact, to take $r=k$ in the definition of $z(t)$ and

$$E=C[0,\tau;Y],\qquad F=C[0,\tau;X],\qquad D(L)=C[0,\tau;D(B_0)],$$
$$D(M)=C[0,\tau;D(B_1)],\qquad (Lu)(t)=B_0u(t),$$
$$(Mv)(t)=B_1v(t)\quad\text{for } u\in D(L),\quad v\in D(M),$$
$$D(B)=\{u\in C^{(1)}[0,\tau;Y]\colon u(0)=0\},\qquad (Bu)(t)=u'(t).$$

Results analogous to Theorem 4.9 and Corollary 4.10 can be obtained for Problem (10) when a spectral hypothesis (on $a$-regions) is satisfied by the operator-pencil $zB_1+B_0$.

EXAMPLE 4.14. Let $a_j(x)$, $b(x)$, $m(x)$, $x\in R^n$, be square $N\times N$ matrices. Following [11, pp. 73–75] for the regular case ($m(x)\equiv I$), we will give in the sequel conditions sufficient to treat the following initial-value problems of partial differential equations, namely

$$\partial(m(x)\,u)/\partial t=\sum_{j=1}^{n}a_j(x)\,\partial u/\partial x_j+(b(x)+\alpha I)\,u+f(t,x),$$
$$0\leqslant t\leqslant\tau,\ x\in R^n, \tag{11}$$
$$\lim_{t\downarrow 0}m(x)\,u(t,x)=w_0(x),\qquad x\in R^n \tag{12}$$

and

$$m(x)\,\partial u/\partial t=\sum_{j=1}^{n}a_j(x)\,\partial u/\partial x_j+(b(x)+\alpha I)\,u+f(t,x),$$
$$0\leqslant t\leqslant\tau,\ x\in R^n, \tag{13}$$
$$u(0,x)=u_0(x),\qquad x\in R^n, \tag{14}$$

where $\alpha$ is a sufficiently large real number and $u=(u_1,...,u_N)$ is the unknown.

The novelty with respect to the treatment given in [11] is that $m(x)$ can be singular, in the sense we will specify in a moment. Let us introduce the space $B^{(i)}(R^n)$, $i=0,1$. It consists of all functions whose derivatives up to order $i$ are continuous and bounded on $R^n$. Then our requests on the matrices in (11), (13) are

each component of $m$ is in $B^0(R^n)$ and $m(x)$ is non-negative for all $x\in R^n$, (15)

each component $a_j^{ik}(x)$, $i$, $k=1,...,N$, $j=1,...,n$, of $a_j(x)$ and $b(x)$ belongs to $B^{(1)}(R^n)$ and $B^{(0)}(R^n)$, respectively; (16)

For any $u \in L^2(R^n)^N = X$, where $X$ is endowed with the usual inner product $(\ ,\ )$, one puts $\mathscr{A}u(x) = \sum_{j=1}^n a_j(x)\, \partial u/\partial x_j + b(x)\, u$.

Define an operator $A$ by

$$D(A) = \{u \in X: \mathscr{A}u \in X\}, \qquad Au = \mathscr{A}u,\ u \in D(A),$$

and $A_0 = A + \alpha I$. Further, let $A_1$ denote multiplication by $m$ in the space $X$. With the notation $u \cdot v = \sum_{i=i}^N u_i \bar{v}_i$ for the inner product of $u = (u_1, ..., u_N)$ and $v = (v_1, ..., v_N)$, it follows from [11, p. 73] that for all $u \in D(A_0) = D(A)$ there holds

$$\operatorname{Re}((A_0 + zA_1)\, u, u) = -\frac{1}{2}\left(u, \sum_{j=1}^n \frac{\partial aj}{\partial x_j} \cdot u\right) + \operatorname{Re}(bu, u) + \alpha \|u; X\|^2.$$

Let

$$\beta = \sup_x \sup_{k,l=1,...,N} |b_{kl}(x)|,$$

$$\tilde{C} = \sup_{j=1,...,n} \sup_x \sup_{k,l=1,...,N} \left|\frac{\partial a_j^{kl}(x)}{\partial x_j}\right|,$$

$$C = n\tilde{C};$$

here, of course, $b_{kl}(x)$ are the components of $b(x)$. Then

$$\operatorname{Re}((A_0 + zA_1)\, u, u) \geqslant (-C/2 - \beta + \alpha)\|u; X\|^2$$

for all complex numbers $z$ such that $\operatorname{Re} z \geqslant 0$. Hence, if $\alpha > \beta + C/2$, we deduce that for a certain $C_1 > 0$ we have

$$\|(A_0 + zA_1)\, u; X\| \geqslant C_1 \|u; X\|, \qquad u \in D(A_0).$$

By introducing the formal adjoint of $\mathscr{A}$, analogously to [11, p. 74], we easily recognize that under the previous hypotheses $zA_1 + A_0$ has a bounded inverse for $\operatorname{Re} z \geqslant 0$ and the norm of $(zA_1 + A_0)^{-1}$ in $L(X)$ is uniformly bounded on this set. Since $A_1$ is a bounded operator, this in turn implies that $\|A_0(zA_1 + A_0)^{-1}; L(X)\| \leqslant C(1 + |z|)$, $\operatorname{Re} z \geqslant 0$.

Then Theorems 4.3, 4.6, 4.12, 4.13 permit the affirmation of

THEOREM 4.15. *Assume* (15), (16). *Then for any* $\alpha$ *sufficiently large and for all* $f \in C^{(3)}[0, \tau; L^2(R^n)^N]$ *there is a unique* $u \in C[0, \tau; L^2(R^n)^N]$, *with* $t \to m(\cdot)\, u(t, \cdot) \in C^{(1)}[0, \tau; L^2(R^n)^N]$ *such that* (11) *holds almost everywhere in* $R^n$ *and* (12) *is verified in the sense that* $\lim_{t \downarrow 0} \int_{R^n} \|m(x)\, u(t, x) - w_0(x); C^N\|^2\, dx = 0$, *provided that* $w_0 = \sum_{k=0}^2 (-1)^j\, (A_1 A_0^{-1})^{j+1} f^{(j)}(0) + w$, *with* $w \in R((A_1 A_0^{-1})^4)$.

Clearly, we have defined $u(t)$, $f(t)$ by $u(t)(x) = u(t, x)$, $f(t)(x) = f(t, x)$, $0 \leqslant t \leqslant \tau$, $x \in R^n$, and $f^{(j)}(t)$ by $f^{(j)}(t)(x) = \partial^j f(t, x)/\partial t^j$, $j = 1, 2, ...$. Note that $w \in R((A_1 A_0^{-1})^s)$, $s=1, 2, ...,$ signifies that $w(x)=m(x)\, v_0(x)$, where $v_0 \in D(A)$ and there exist $v_1, ..., v_{s-1} \in D(A)$ such that $((A+\alpha)\, v_i)(x) = m(x)\, v_{i+1}(x)$, for $i=0, 1, ..., s-2$.

THEOREM 4.16. *Under the same hypotheses as in Theorem* 4.15, *there is a unique* $u$ *solving* (11), (12) *for all* $f$ *and* $w_0$ *such that* $f(t, \cdot) = (A_1 A_0^{-1})^3\, g(t, \cdot)$, $w_0 = (A_1 A_0^{-1})^4\, w_1$, *where* $g \in C[0, \tau; L^2(R^n)^N]$ *and* $w_1 \in L^2(R^n)^N$.

THEOREM 4.17. *If the asumptions in Theorem* 4.15 *hold, and* $f \in C^{(4)}[0, \tau; L^2(R^n)^N]$, $u_0 \in D(A)$, *then for* $\alpha$ *sufficiently large* (*independent of* $f$ *and* $u_0$), *there exists a unique* $u \in C[0, \tau; L^2(R^n)^N]$, *with* $t \to u(t, \cdot) \in C^{(1)}[0, \tau; L^2(R^n)^N]$ *such that* (13) *is satisfied almost everywhere in* $R^n$ *and* $\lim_{t \downarrow 0} \int_{R^n} \|u(t, x) - u_0(x); C^N\|^2\, dx = 0$, *provided that*

$$(A+\alpha)\, u_0 - \sum_{j=0}^{3} (-1)^j\, (A_1(A+\alpha)^{-1})^j f^{(j)}(0) \in R((A_1(A+\alpha)^{-1})^4).$$

The same conclusion holds if $f(t, \cdot) = (A_1(A+\alpha)^{-1})^4\, g(t, \cdot)$, $(A+\alpha)\, u_0 \in R((A_1(A+\alpha)^{-1})^4)$, $g \in C[0, \tau; L^2(R^n)^N]$.

*Remark* 4.18. In [3], R. Beals proved existence and uniqueness of solutions to the Cauchy problem and mixed problems for a general class of (possibly) non-strictly hyperbolic equations and systems, by reducing them to the abstract form (6).

In the Cauchy problem case, his assumptions imply that if $X = L^2(R^n)^N$ and the matrices involved in (13), with $m(x) \equiv I$, are independent of $x$, then the estimate $\|(z+A)^{-1}; L(X)\| \leqslant C(1+|z|)^m$, $m$ a suitable positive integer, holds on $|\mathrm{Re}\, z| \geqslant c_0 > 0$. On the other hand, if the matrices in (13) do depend on $x$, then such an inequality is satisfied in a region of type $a$. Analogous results hold for mixed problems. Hence, we can apply both Theorems 4.3, 4.6, together Remark 4.5, in the former case, and Corollary 4.10 in the latter one.

*Application* 2

Consider the following Cauchy problem relative to a higher order differential equation,

$$A_n u^{(n)}(t) + \cdots + A_j u^{(j)}(t) + \cdots + A_1 u'(t) + A_0 u(t) = f(t), \qquad 0 \leqslant t \leqslant \tau,$$
$$u(0) = u_0,\, u'(0) = u_1, ..., u^{(n-1)}(0) = u_{n-1}, \tag{17}$$

where $A_j$, $j=0, 1, ..., n$, is a closed linear operator from $X$ into itself, $X$ a complex Banach space, $f \in C[0, \tau; X]$ and $u_j \in X$ for $j=0, 1, ..., n-1$. Clearly, $u^{(j)}(t)$ denotes the strong derivative of order $j$ of $u(t)$: $u^{(j)}(t) =$

$d^j u(t)/dt^j$. As a *strict* solution for (17) we mean a function $u \in C^{(n)}[0, \tau; X]$ such that $u^{(j)}(t) \in D(A_j)$ for all $0 \leqslant t \leqslant \tau$ and $j = 0, 1, ..., n$, $t \to A_j u^{(j)}(t)$ belongs to $C[0, \tau; X]$ and (17) holds.

In the sequel we will give some hypotheses on $A_j$ which ensure that the results in Application 1 apply to a suitable first order problem to which (17) is reduced. To this end, we shall make

ASSUMPTION H.10. *$A_0$ and $Q(z) = \sum_{j=0}^{n} z^j A_j$ have a bounded inverse for all $z$ in a region $\mathscr{B}$, where $\mathscr{B}$ is either a logarithmic region or a region of type a. Moreover, there exist $q_j \geqslant 0$, $j = 0, 1, ..., n$ and $C > 0$, such that*

$$\|A_j Q(z)^{-1}; L(X)\| \leqslant C(1 + |z|)^{q_j}, \qquad z \in \mathscr{B},\ j = 0, 1, ..., n.$$

Now define

$$\tilde{q} = \max\{q_j,\ j = 0, 1, ..., n\},$$

$$X_j = D(A_{j+1}) \cap D(A_{j+2}) \cap \cdots \cap D(A_n), \qquad j = 0, 1, ..., n-2,$$

where it is supposed that $X_j \neq \{0\}$; $X_j$ is endowed with the norm $\|u; X_j\| = \max\{\|u; D(A_{j+r})\|: r = 1, 2, ..., n-j\}$,

$$X_{n-1} = X, \qquad Z = X_0 \times X_1 \times \cdots \times X_{n-1},$$

$$C_j(z) = \sum_{r=j+1}^{n} z^{r-j-1} A_r, \qquad D(C_j(z)) = X_j, \qquad j = 0, 1, ..., n-2,$$

$C_{n-1}(z) = I$, the identity operator,

$$D_j(z) = \sum_{r=j+1}^{n-1} z^{r-j-1} A_r, \qquad D(D_j(z)) = \bigcap_{r=j+1}^{n-1} D(A_r), \quad j = 0, 1, ..., n-2,$$

$$D_{n-1}(z) = 0, \qquad T(z) = \sum_{j=0}^{n-1} z^j A_j.$$

Let us set $u^{(j)}(t) = v_j(t)$, $j = 0, 1, ..., n-1$, $v(t) = (v_0(t), ..., v_{n-1}(t))$, $v_0 = (u_0, ..., u_{n-1})$, $X^n = X \times \cdots \times X$, $F(t) = (0, 0, ..., 0, f(t))$, and define two operators $\mathscr{L}$ and $\mathscr{M}$ in $X^n$ by

$$\mathscr{L}v = \left(-v_1, -v_2, ..., -v_{n-1}, \sum_{j=0}^{n-1} A_j v_j\right),$$

$$D(\mathscr{L}) = D(A_0) \times D(A_1) \times \cdots \times D(A_{n-1}),$$

$$\mathscr{M}w = (w_0, w_1, ..., w_{n-2}, A_n w_{n-1}),$$

$$w = (w_0, ..., w_{n-1}) \in D(\mathscr{M})$$

$$= \underbrace{X \times \cdots \times X}_{n-1} \times D(A_m).$$

Then (17) assumes the form

$$\mathcal{M}v'(t) + \mathcal{L}v(t) = F(t), \qquad 0 \leqslant t \leqslant \tau,\ v(0) = v_0. \tag{18}$$

The crucial problem now is to find a suitable space $E$ in which to apply to (18) the technique we have developed in Application 1. We begin by observing that $(z\mathcal{M} + \mathcal{L})\, v = h = (h_0, h_1, ..., h_n)$ is equivalent to the system

$$zv_j - v_{j+1} = h_j, \qquad j = 0, 1, ..., n-2,$$
$$\sum_{j=0}^{n-2} A_j v_j + (zA_n + A_{n-1})\, v_{n-1} = h_{n-1}.$$

In view of Assumption H.10, we deduce that for all $z \in \mathcal{B}$, one has

$$v_0 = Q(z)^{-1} \sum_{j=0}^{n-1} C_j(z)\, h_j,$$
$$v_j = z^j v_0 - \sum_{r=0}^{j-1} z^{j-1-r} h_r, \qquad j = 1, ..., n-1;$$

hence, we also have

$$v_j = \sum_{r=0}^{j-1} (z^j Q(z)^{-1} C_r(z) - z^{j-1} I)\, h_r$$
$$+ \sum_{r=j}^{n-1} (z^j Q(z)^{-1} C_r(z))\, h_r, \qquad j = 1, 2, ..., n-1.$$

Some cumbersome calculations then show that $(z\mathcal{M} + \mathcal{L})^{-1} f = x = (x_1, ..., x_n)$, with $f = (f_1, ..., f_n) \in Z$, if and only if

$$x_k = \sum_{j=1}^{k} z^{k-j}(I - z^j Q(z)^{-1} C_{j-1}(z)) f_j$$
$$- z^k Q(z)^{-1} \sum_{j=k+1}^{n} C_{j-1}(z) f_j \qquad \text{for} \quad k = 1, ..., n-1,$$
$$x_n = \sum_{j=1}^{n} (T(z)\, Q(z)^{-1} C_{j-1}(z) - D_{j-1}(z)) f_j.$$

Note that $x_k \in X_{k-1}$ for $k = 1, ..., n-1$. What we have just obtained and the estimates we will prove in a moment, shall show that $E = C[0, \tau; Z]$ works. In view of the estimate ($M$ is a suitable constant)

$$\|C_j(z); L(X_j; X)\| \leqslant M(1 + |z|)^{n-1-j}, \qquad j = 0, 1, ..., n-2,$$

since $f_j \in D(A_k)$ for $j = 0, 1, ..., k$, it is a simple matter to recognize that

$$\|x_k; X\| \leqslant C(1+|z|)^{q_0+k+n-1} \sum_{j=1}^{n} \|f_j; X_{j-1}\|$$
$$= C(1+|z|)^{q_0+k+n-1} \|f; Z\|,$$
$$\|A_j x_k; X\| \leqslant C'(1+|z|)^{q_j+k+n-1} \|f; Z\|, \qquad j = k, k+1, ..., n,$$

and hence

$$\|x_k; X_{k-1}\| \leqslant C''(1+|z|)^{\tilde{q}+k+n-1} \|f; Z\|, \qquad k = 1, 2, ..., n-1.$$

Further, the estimates ($C_i$, $i = 1, 2, ...$, are suitable positive constants):

$$\|T(z)\, Q(z)^{-1}; L(X)\| \leqslant C_1(1+|z|)^s, \; s = \max\{q_0, q_1+1, ..., q_{n-1}+n-1\},$$
$$\|D_j(z); L(X_j; X\| \leqslant C_2(1+|z|)^{n-2-j}, \qquad j = 0, 1, ..., n-2,$$

imply that

$$\|x_n; X\| \leqslant C_3(1+|z|)^{s+n-1} \|f; Z\|.$$

On the other hand, $s \leqslant \tilde{q}+n-1$, and thus

$$\|x; Z\| \leqslant C_4(1+|z|)^{\tilde{q}+2(n-1)} \|f; Z\|.$$

Therefore all we have obtained in Application 1 is applicable with $\tilde{q}+2(n-1)$ instead of $m$. We could then make use of Theorems 4.12, 4.13 (and their analogues for $a$-regions) and reach regularity and compatibility conditions on $f$ and the initial data $u_0, ..., u_{n-1}$, which ensure solvability for (17).

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